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# On the stability and over-reflexion of hydromagnetic–gravity waves

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The stability of a magnetic-velocity shear layer, of thickness $L$, to hydromagnetic–gravity waves of zonal wavenumber $k$ is investigated analytically, within the Boussinesq approximation, in the situation where $\epsilon(=kL)$ is small. It is found that, in addition to the unstable modes of the corresponding sheet, new modes of instability of growth rate of order $\epsilon^2$ are also present provided *one* critical level exists within the layer. The existence of one critical level also effects over-reflexion of stable modes. Furthermore it is shown that a magnetic shear acting alone can lead to instability as well as effecting over-reflexion of stable modes.

## 1. Introduction

The stability and propagation properties of wave motions in magnetic shear or velocity shear or both are relevant to many geophysical, astrophysical and engineering problems (see, for example, Drazin & Howard 1966; Dickinson 1968; Baldwin & Roberts 1970, 1972; Moffatt 1976; Drazin & Davey 1977; El Mekki, Eltayeb & McKenzie 1978). However, because of the lack of analytical solutions for general flow profiles or general field profiles, most stability studies have been confined to vortex sheets or current sheets (see, for example, Miles 1957; Fejer 1964; McKenzie 1970; Grimshaw 1976; Acheson 1976) on the belief that the sheet treatment represents an adequate approximation to a thin shear layer (in the sense that the thickness of the layer is much smaller than the wavelength of the relevant waves). Such an approximation has been confirmed for gravity waves in a Boussinesq fluid subject to a linear velocity shear by Eltayeb & McKenzie (1975). On the other hand, Blumen, Drazin & Billings (1975) in their study on the stability of an inviscid compressible shear layer isolated modes of instability which have no counterparts in the corresponding vortex sheet treatment.

The problem of reflexion of waves by a vortex sheet or current sheet has also been generally accepted as providing a reasonable representation of a thin shear layer. However, most of the thoroughly studied vortex and current sheets predicted over-reflexion in the circumstances in which at least one critical level is embedded within the sheet. On the other hand, recent studies on the reflectivity of various types of waves by magnetic-velocity shear (Eltayeb 1977; Eltayeb & McKenzie 1977; El Sawi & Eltayeb 1978) have demonstrated that the presence of critical levels can *enhance* over-reflexion.

The above discussion poses the following questions. Do the properties of thin shear

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layers generally differ from those of the corresponding sheet and, if so, what are the special cases, if any, which match uniformly with the sheet? What influence does the shear profile play in the stability and reflectivity properties of thin shear layers? What effect do critical levels, within the context of the dissipationless theory, have on these properties? The purpose of the present study is to shed some light on the answers to these questions.

Acheson (1976) made a detailed study of the reflectivity and stability of a current-vortex sheet in a Boussinesq fluid. In this paper we replace Acheson's sheet by a thin shear layer, of thickness $L$. Adopting $\epsilon$ $(=kL$, where $k$ is the zonal wavenumber) as a small parameter, we develop an expansion scheme in $\epsilon$ to study the stability, reflectivity and transmissivity of the shear. For general flow and field profiles within the shear, the stability of the system to order $\epsilon$ is found to be governed by the same conditions governing the stability of the current-vortex sheet. This result hinges on the fact that to leading order of approximation the solution within the shear layer can adjust itself in such a manner that it is regular everywhere and therefore unaffected by the presence of critical levels. When higher-order terms are considered, however, the presence of critical levels which appear at order $\epsilon$ in the solution within the layer is shown to change the stability of the system. Since the influence of the *thin* shear on the frequency $\omega$ of the waves is order $\epsilon^2$, attention is focused on the neutral modes of the current-vortex sheet with the purpose of investigating (analytically) the possibility of the occurrence of new modes of instability due to the presence of the shear. If only one critical level exists within the shear a mode of instability which is absent in the current-vortex sheet is present here and has a growth rate of the order $\epsilon^2$.

The reflexion of hydromagnetic–gravity waves by the shear is also studied. It is found that the presence of only one critical level within the thin shear also effects over-reflexion. However, the over-reflecting and unstable régimes due to the presence of the same critical level involve different modes. This is demonstrated by the analytical results of a linear magnetic shear in the absence of a flow where over-reflexion is present in a stable magnetic shear.

In § 2 we define the problem; in § 3 we study the linear shear models; in § 4 we discuss the general magnetic-velocity shear and in § 5 some concluding remarks are made.

## 2. Definition of the problem

Consider a Boussinesq fluid in the presence of gravity. Suppose that it is flowing with a velocity $\mathbf{U}$ in the presence of an Alfvén velocity $\mathbf{V}$, where

$$(\mathbf{U}, \mathbf{V}) = (U, V)\,\hat{\mathbf{x}}, \tag{2.1}$$

$$U, V = \left\{\begin{array}{lll} U_1, V_1, & \text{for} \quad 0 \geqslant z & \text{(region 1)}, \\ U(z), V(z) & \text{for} \quad 0 \leqslant z \leqslant L & \text{(region 2)}, \\ U_3, V_3 & \text{for} \quad L \leqslant z & \text{(region 3)}. \end{array}\right\} \tag{2.2}$$

Here $\hat{\mathbf{x}}$ is a unit vector along the horizontal axis $0x$ and $0z$ is the upward vertical co-ordinate axis. We shall assume that $U$ and $V$ are continuous at both $z = 0, L$. The state (2.1), (2.2) is consistent with the basic equations of motion, induction and continuity provided the total pressure is in hydrostatic equilibrium (cf. Eltayeb 1977, § 4). If we superimpose perturbations of the normal mode type

$$f(x, z, t) = \mathrm{Re}\{f(z) \exp i(\omega t - kx)\}, \tag{2.3}$$

on the above basic state, linearize the equations of motion, induction and continuity, adopt the Boussinesq approximation, and eliminate all variables in favour of the vertical component of velocity, $w$, we obtain

$$Pw'' + P'w' + \{k(PU'' + P'U')/\hat{\omega} - Pk^2 - k^2N^2/\hat{\omega}^2\} w = 0, \tag{2.4}$$

in which the intrinsic (Doppler-shifted) frequency $\hat{\omega}$ and $P$ are defined by

$$\hat{\omega} = \omega - kU, \quad P = -1 + k^2V^2/\hat{\omega}^2, \tag{2.5}$$

and $N$ is the usual Brunt–Väisälä (buoyancy) frequency. The perturbation in total pressure, $p$, which we will require later is

$$p = iP(\hat{\omega}w' + kU'w)/k^2. \tag{2.6}$$

The accent in (2.4) and (2.6) denotes differentiation with respect to the argument.

We find it convenient in the analysis below to make the transformation

$$w = P^{-\frac{1}{2}}\phi \tag{2.7}$$

so that (2.4) assumes the normalized form

$$\phi'' + g(z)\,\phi = 0, \tag{2.8}$$

in which

$$g(z) = \frac{k\omega U'' + k^2(V'^2 + VV'')}{\hat{\omega}^2 - k^2V^2} - k^2\left(1 - \frac{N^2}{\hat{\omega}^2 - k^2V^2}\right) + \frac{k^3V(kVV'^2 + 2\hat{\omega}V'U' + kVU'^2}{(\hat{\omega}^2 - k^2V^2)^2}. \tag{2.9}$$

The boundary conditions associated with the system are the continuity of $w$ and $p$ at both $z = 0, L$. Thus

$$[w] = 0, \quad [\hat{\omega}w' + kU'w] = 0, \tag{2.10}$$

where the square brackets denote the jump in the quantity within. In addition the solutions must satisfy extra conditions in regions 1 and 3. Suppose a wave of given amplitude $I$ is incident on the shear from region 1. Then the solution can be written as

$$w = \left\{\begin{array}{ll} I\exp(im_1z) + R\exp(-im_1z) & \text{in region 1,} \\ Aw_1(z) + Bw_2(z) & \text{in region 2,} \\ T\exp(im_3z) & \text{in region 3,} \end{array}\right\} \tag{2.11}$$

where $w_1$ and $w_2$ are any two independent solutions of (2.4) in region 2, and $m_1$ and $m_3$ are governed by the dispersion relation

$$m^2 = k^2[-1 + N^2/(\hat{\omega}^2 - k^2V^2)], \tag{2.12a}$$

when $U$ and $V$ take the values appropriate to regions 1 and 3. When the wavenumbers $m_1$ and $m_3$ are real, they must be chosen such that the incident wave transports energy *toward* the shear layer while the reflected and transmitted waves transport energy

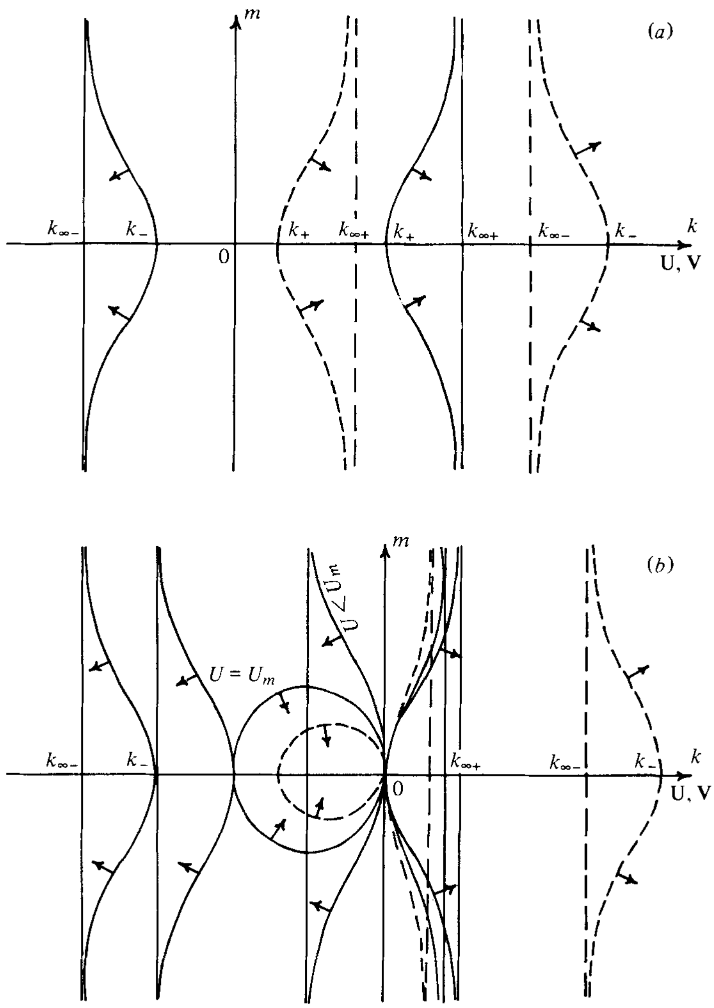


FIGURE 1. The wave-normal curves for hydromagnetic–gravity waves in a Boussinesq fluid. (a) $N^2 < \omega^2$, (b) $N^2 > \omega^2$. In both (a) and (b)

$$k_{\infty\pm} = \omega/(U \pm V), \quad k_{\pm} = \{\omega U \pm [\omega^2 V^2 + N^2(U^2 - V^2)]^{\frac{1}{2}}\}/(U^2 - V^2)$$

and in (b) $U_m = (1 - \omega^2/N^2)^{\frac{1}{2}} V$. The solid (broken) curves refer to $U < V$ $(U > V)$. The arrows denote the direction of the group velocity. The figure is drawn for $\omega$, $U$, $V > 0$.

*away* from the layer. The direction of propagation of energy flux is determined by the group velocity (Lighthill 1965). The energetics of the present system have been studied in detail by Acheson (1976). The component of group velocity in the vertical direction is given by

$$\partial\omega/\partial m = -mk^2N^2/\hat{\omega}(m^2+k^2)^2, \tag{2.12b}$$

so that the radiation condition ($m_3$ real) is determined by choosing $m_3\hat{\omega}_3 < 0$. The wave-normal curves and the group velocity are summarized in figure 1. Now, if either $m_1$ or $m_3$ is imaginary, it must be chosen such that the solution decays as $|z|$ tends to infinity.

For the problem of reflexion, $m_1$ is always real, but $m_3$ may be real or imaginary, while for the stability problem no wave is incident on the layer (i.e. $I = 0$) and in general both $m_1$ and $m_3$ may take real or imaginary values.

Our interest here lies in the problem of the thin shear layer (or long waves) so that $|kL| \ll 1$. It is then natural to employ the quantity

$$\epsilon = kL \tag{2.13}$$

as a small parameter in an expansion scheme. Thus we let

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots, \tag{2.14a}$$

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots, \tag{2.14b}$$

$$R = R_0 + \epsilon R_1 + \epsilon^2 R_2 + \ldots, \tag{2.14c}$$

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \ldots, \tag{2.14d}$$

so that

$$g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \ldots, \tag{2.15a}$$

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2, \tag{2.15b}$$

$$\overline{m} = \overline{m}^{(0)} + \overline{m}^{(1)}\epsilon + \overline{m}^{(2)}\epsilon^2 + \ldots, \tag{2.15c}$$

where

$$\overline{m} = m/k. \tag{2.16}$$

We now substitute (2.13)–(2.16) into (2.7)–(2.12) and equate the coefficients of $\epsilon^n$ ($n = 0, 1, 2, \ldots$) in each resulting equation to zero to obtain a hierarchy of sets of equations which can be solved successively. This will be carried out in the next two sections.

Since we are interested in comparing the present problem with the corresponding sheet treatment of Acheson (1976), we shall require $U$ or $V$ or both to vary appreciably over the thin shear no matter how thin it may be. For this purpose $U$ and $V$ must take the form

$$U = U_1 + U_0 F(Z), \tag{2.17a}$$

$$V = V_1 + V_0 H(Z), \tag{2.17b}$$

where

$$U_0 = U_3 - U_1, \quad V_0 = V_3 - V_1, \quad Z = z/L, \quad F(0) = H(0) = 0, \quad F(1) = H(1) = 1. \tag{2.18}$$

## 3. Stability and reflectivity of a linear shear

In this section we first assume that $V$ is uniform, but necessarily non-zero, everywhere and $U$ is linearly sheared in region 2. Thus

$$V_0 = 0, \quad F(Z) = Z. \tag{3.1}$$

Since $V$ occurs only in the form $V_1^2$ we assume, without any loss of generality, that

$$V_1 = V > 0. \tag{3.2}$$

It transpires that considerable simplifications occur in the present problem if we introduce the quantity $X$ defined by

$$X = \hat{\omega}/kV, \quad \hat{\omega} = \omega_0 - kU. \tag{3.3}$$

Following the procedure outlined in the penultimate paragraph of § 2 above we now proceed to solve the stability and reflectivity problems.

### 3.1. *The stability analysis*

Here the solution is given by (2.11) with $I$ set equal to zero. We will find it sufficient to consider the three problems $n = 0, 1, 2$ in order to determine the stability of the system to leading order and it is convenient to consider the successive problems separately.

*Problem* 0. The zeroth-order problem (i.e. $n = 0$) is

$$\phi_0''+g_0\phi_0 = 0, \quad g_0 = (X^2-1)^{-2}, \quad w_0 = \phi_0 X/(1-X^2)^{\frac{1}{2}}, \tag{3.4}$$

subject to the boundary conditions

$$[w_0] = 0, \quad [w_0 - Xw_0'] = 0 \quad \text{at} \quad X = X_1, X_3, \tag{3.5}$$

where $X_1$ and $X_3$ are the values of $X$, as defined by (3.3), at $Z = 0, 1$ respectively, and the accent denotes differentation with respect to the argument ($X$).

The second-order equation (3.4) can be solved formally by transforming it into the (degenerate) hypergeometric equation and using standard textbooks. Omitting the details, the two linearly independent solutions can be written simply as

$$\phi_0^{(1)} = (1-X^2)^{\frac{1}{2}}, \quad \phi_0^{(2)} = (1-X^2)^{\frac{1}{2}}\ln\chi, \quad \chi = (X+1)/(X-1), \tag{3.6}$$

so that
$$w_0^{(1)} = X, \quad w_0^{(2)} = X\ln\chi. \tag{3.7}$$

Now the boundary conditions (3.5) give

$$R_0 = w_0(X_1), \quad T_0 = w_0(X_3), \tag{3.8}$$

$$0 = w_0(X_1) - X_1 w_0'(X_1), \tag{3.9}$$

$$0 = w_0(X_3) - X_3 w_0'(X_3), \tag{3.10}$$

where
$$w_0 = A_0 w_0^{(1)} + B_0 w_0^{(2)}. \tag{3.11}$$

It then immediately follows that

$$B_0 = 0, \quad T_0 = A_0 = A_0 X_3, \quad R_0 = T_0 X_1/X_3. \tag{3.12}$$

Thus, to leading order, the solution within the shear is regular everywhere whether critical levels (occurring where $X^2 = 1$) are present or not. Moreover, the expressions (3.12) for the amplitudes $R$, $T$, $A_0$ and $B_0$ are not sufficient to determine the stability criterion and it is necessary to proceed to the next problem.

*Problem* 1. When $n = 1$, we have

$$\phi_1''+g_0\phi_1 = -g_1\phi_0, \tag{3.13}$$

with
$$g_1 = \frac{-4\tilde{\omega}_1 X}{(X^2-1)^3}, \quad w_1 = \frac{X\phi_1}{(1-X^2)^{\frac{1}{2}}} + \frac{\tilde{\omega}_1 w_0}{X^2(1-X^2)}, \tag{3.14}$$

and the boundary conditions take the form

$$R_1 = w_1(X_1), \quad T_1 + i\overline{m}_3^{(0)}\, T_0 = w_1(X_3), \tag{3.15}$$

$$-i\overline{m}_1^{(0)}\,\alpha^{-1}R_0 = w_1'(X_1) - X_1^{-1}w_1(X_1) + \tilde{\omega}_1 X_1^{-1} w_0(X_1), \tag{3.16}$$

$$i\overline{m}_3^{(0)}\,\alpha^{-1}T_0 = w_1'(X_3) - X_3^{-1}w_1(X_3) + \tilde{\omega}_1 X_3 w_0(X_3), \tag{3.17}$$

where
$$\tilde{\omega}_1 = \omega_1/kV, \quad \alpha = U_0/V. \tag{3.18}$$

From the theory of differential operators, it is well known that, for (3.13) to possess a solution, the right-hand side must be orthogonal to the solution of the adjoint homogeneous equation. Noting that the homogeneous part is identical with (3.4) and using the elementary complex inner product (since $\phi_0$ is complex), we find that the necessary solvability condition is obtained by multiplying (3.13) by $\phi_0^*$ (the complex conjugate of $\phi_0$) and integrating between $X_1$ and $X_3$. Thus

$$4\tilde{\omega}_1 \int_{X_1}^{X_3} \frac{X\,|\phi_0|^2\,dX}{(X^2-1)^3} = i\alpha^{-1}A_0^*\,[\overline{m}^{(0)}R_0(1-X_1^2)/X_1 + \overline{m}_3^{(0)}\,T_0(1-X_3^2)/X_3] + \tilde{\omega}_1\,|A_0|^2 h_1, \qquad (3.19)$$

where

$$h_1 = \frac{3X_1^2-1}{X_1^2(1-X_1^2)} - \frac{3X_3^2-1}{X_3^2(1-X_3^2)} - \frac{(1-X_3^2)}{X_3^2} + \frac{1-X_1^2}{X_1^2}. \qquad (3.20)$$

The integral on the left-hand side of (3.19) is convergent if the interval $[X_1, X_3]$ does not include any of the singularities $X = \pm 1$. If, however, one of the two singularities lies in that interval, then the integral should be interpreted as the Cauchy principal value of the integral which can be obtained by contour integration along a path suitably dented at the singularities (see the appendix). It can be shown, without going into details here, that the Cauchy principal value exists for all the cases required here. Such a procedure gains support from the detailed study of the critical layer for gravity waves in a shear flow (Baldwin & Roberts 1970) in the presence of weak viscous diffusion which yields the same results obtained by Booker & Bretherton (1967) for the corresponding inviscid model. (The inclusion of diffusion in the present model renders the problem extremely difficult and its treatment is outside the scope of the present note (see Baldwin & Roberts 1972).) The same procedure applies to the integral on the left-hand side of (3.34).

Provided (3.19) is obeyed, (3.13) can be solved in terms of complementary and particular integrals, and since the complementary function is known (cf. (3.4)), the general solution can be written down explicitly. However, for our purposes here it is sufficient to write it in the form

$$w_1 = A_1 X + B_1 X \ln\chi + \tilde{\omega}_1 w_p. \qquad (3.21)$$

Now application of (3.16) and (3.17) to (3.21) yields

$$-i\alpha^{-1}[\overline{m}_1^{(0)}R_0(1-X_1^2)/X_1 + \overline{m}_3^{(0)}T_0(1-X_3^2)/X_3] = \tilde{\omega}_1\left\{\frac{(1-X_1^2)}{X_1}[w_p'(X_1) - w_p(X_1)/X_1 + A_0] + \frac{(1-X_3^2)}{X_3}[w_p'(X_3) - w_p(X_3)/X_3 + A_0]\right\}. \qquad (3.22)$$

From (3.20) and (3.22) we deduce that

$$\tilde{\omega}_1 = 0, \qquad (3.23)$$

and, after using the last of (3.12),

$$\overline{m}_1^{(0)}(1-X_1^2) + \overline{m}_3^{(0)}(1-X_3^2) = 0. \qquad (3.24)$$

This last equation gives the normal modes of the system which must be solved together with the appropriate conditions on $m_1$ and $m_3$. It is to be noted that, despite appearances, this equation is identical to that obtained for the corresponding sheet treatment

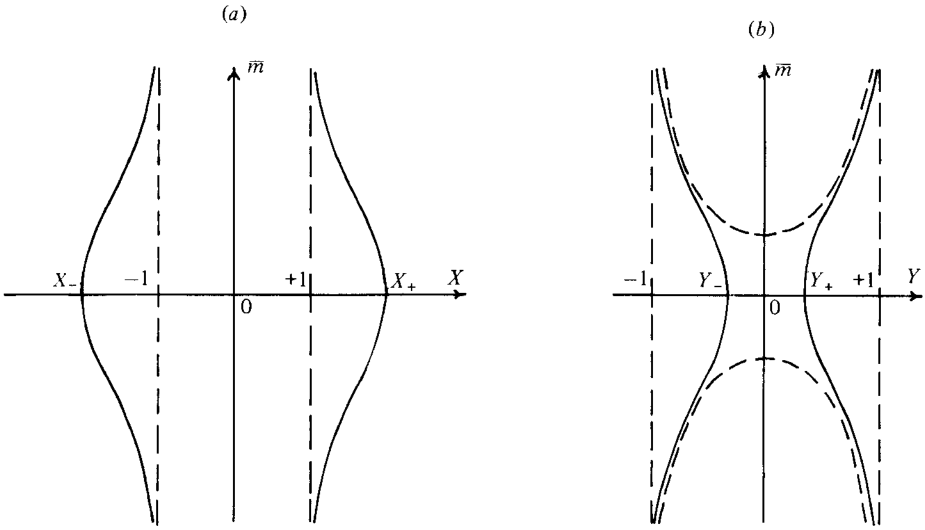


FIGURE 2. Geometric representation of the relations (3.25) and (3.49). Here $X_\pm = \pm(1+\beta^2)^{\frac{1}{2}}$, $Y_\pm = \pm(1-\beta_1^2)^{\frac{1}{2}}$, considered to be both real. The straight discontinuous lines refer to $X = \pm 1$, $Y = \pm 1$. In (*b*), the solid (broken) curves refer to $\beta_1^2 < 1$ ($\beta_1^2 > 1$).

by Acheson (1976). Now (3.24) shows that the natural modes (i.e. $X_1$ and $X_3$ real) must occur for $\overline{m}_1^{(0)}/\overline{m}_3^{(0)}$ real, so that $m_1$ and $m_3$ are either both real and the shear will radiate on either side or both imaginary and the solutions will decay exponentially on either side of the shear. This is particularly evident when we express the dispersion relation (2.12) in the form

$$\overline{m}^2 = -1+\beta^2/(X^2-1), \quad \beta^2 = N^2/k^2V^2 \tag{3.25}$$

(see figure 2). Evidently $\overline{m}$ is real only if

$$1 < X^2 < 1+\beta^2. \tag{3.26}$$

By using the relation

$$X_3 = X_1+\alpha, \tag{3.27}$$

the roots of (3.24) occur when $X_1$ takes the values

$$X_{1c} = -\tfrac{1}{2}\alpha, \quad X_{1c\pm} = -\tfrac{1}{2}\alpha \pm (1+\tfrac{1}{2}\beta^2-\tfrac{1}{4}\alpha^2)^{\frac{1}{2}}, \tag{3.28}$$

which are the same as those obtainable for the corresponding sheet treatment. The mode $X_{1c}$ satisfies $X_{3c} = -X_{1c}$ and with the help of figure 2*a* and the radiation condition we see that such a mode can occur for real $m_1$ and $m_3$ only if the shear contains *two* critical levels and for imaginary $m_1$ and $m_3$ if either two or no critical levels lie within the shear layer. On the other hand, the modes $X_{1c\pm}$ can occur if one critical level lies inside the shear and both $m_1$ and $m_3$ are imaginary. Thus the modes (3.28) are identical to those of the sheet model because the influence of any critical levels within the shear is not felt to order $\epsilon$. However, when higher-order terms are taken into account the situation is different. Indeed when we solve (3.15)–(3.17) together with (3.23) we find that

$$\left.\begin{aligned} B_1 &= i\alpha^{-1}\overline{m}_3^{(0)}T_0(1-X_3^2)/2X_3, \\ A_1 &= R_1/X_1 - B_1 \ln \chi_1, \\ R_1/X_1 &= (T_1+i\overline{m}_3^{(0)}T_0)/X_3 + B_1 \ln(\chi_1/\chi_3), \end{aligned}\right\} \tag{3.29}$$

so that $B_1 \neq 0$ and the influence of any critical levels present within the shear layer is bound to be felt at the next-order terms.

*Problem* 2. The order-$\epsilon^2$ terms give (using the fact that $\tilde{\omega}_1 = g_1 = \overline{m}^{(1)}_{1,3} = 0$)

$$\phi_2'' + g_0 \phi_2 = -g_2 \phi_0, \tag{3.30}$$

$$g_2 = -\frac{4\tilde{\omega}_2 X}{(X^2-1)^3} + \alpha^{-2}\left[-1 + \frac{\beta^2}{(X^2-1)}\right], \tag{3.31}$$

$$w_2 = \frac{X\phi_2}{(1-X^2)^{\frac{1}{2}}} - \frac{\tilde{\omega}_2 w_0}{X^2(1-X^2)}. \tag{3.32}$$

The boundary conditions are

$$[w_2] = 0, \quad [w_2' + \tilde{\omega}_2 w_0' - X^{-1} w_2] = 0, \tag{3.33}$$

and $w_2$ is obtained from (3.21) by replacing the subscript 1 by 2.

The consistency condition (see problem 1 above) for (3.30) gives

$$-\int_{X_1}^{X_3} g_2 |\phi_0|^2 \, dX = -\tilde{\omega}_2 h_1 |A_0|^2 + i\alpha^{-1} A_0^* [\overline{m}_1^{(0)} R_1 (1 - X_1^2)/X_1 + \overline{m}_3^{(0)} (T_1 + i\overline{m}_3^{(0)} T_0)(1 - X_3^2)/X_3], \tag{3.34}$$

where $h_1$ is given by (3.20). If we now use (3.29) and (3.31) we obtain

$$\tilde{\omega}_2 = C/D, \tag{3.35}$$

in which

$$\left.\begin{aligned} C &= -\tfrac{1}{2}\delta \overline{m}_1^{(0)} (1 - X_1^2)^2 - (1 + \beta^2) + \tfrac{1}{3}(X_3^2 + X_1 X_3 + X_1^2), \\ D &= \frac{2}{1 - X_3^2} - \frac{2}{1 - X_1^2}, \\ \delta &= \ln(\chi_3/\chi_1). \end{aligned}\right\} \tag{3.36}$$

The expression $\delta$ must be interpreted here as containing the appropriate phase jump at every singularity occurring where $X^2 = 1$ (if any). Now for the linear shear considered above the shear layer can contain either (i) no critical levels or (ii) one critical level or (iii) two critical levels. If no critical levels exist then $X_3$ and $X_1$ have the same sign and consequently $\delta$ is real and hence $\tilde{\omega}_2$ is also real. In case (iii) it can be shown (see the appendix) that no net phase jump occurs and again $\tilde{\omega}_2$ is real. Thus in both cases (i) and (iii) the presence of the shear only serves to modify the frequency of the neutral modes and does not lead to any instability, at least to order $\epsilon^2$. If, however, only one critical level exists within the shear then a *net* phase jump is present and $\tilde{\omega}_2$ is necessarily complex. Indeed it can be shown that if $|U_3| > |U_1|$ then the phase jump in $\delta$ is $-\pi$.

As we pointed out in the introduction, our interest here lies mainly in the stability of the neutral modes (3.28), when the quantity within the square root sign is positive, and the influence of the shear on them. Since the mode $X_{1c}$ occurs when $X_3 = -X_1$ (and with the help of figure 2) we see that such a mode can only occur if the layer contains either no critical levels or two of them, in which case $\tilde{\omega}_2$ is real and the presence of the shear only modifies its frequency. In fact it can easily be deduced that all $\tilde{\omega}_n$ ($n = 1, 2, 3, \ldots$) are real for this mode. Thus this mode is always neutrally stable according to the dissipationless theory. (Incidentally this mode is essentially the same neutral mode of instability which gave resonant over-reflexion in the absence of the magnetic field (Eltayeb & McKenzie 1975).) The modes $X_{1c\pm}$, on the other hand, are different and it is

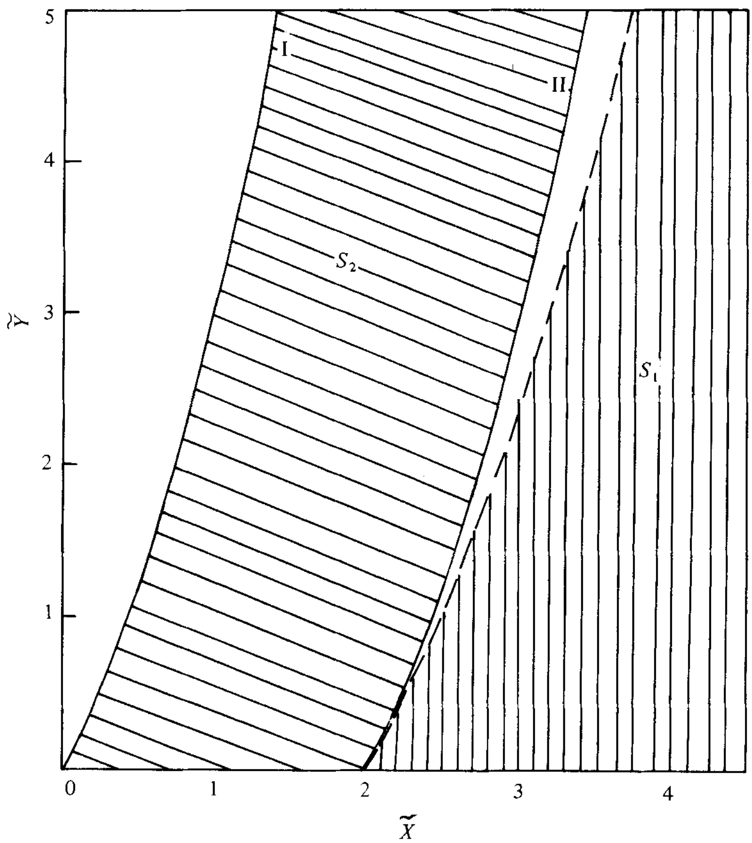


FIGURE 3. The regions of instability of the various unstable modes of the linear velocity shear model in the presence of a non-zero Alfvén speed (see § 2.1). The curves I and II are

$$\tilde{Y} = 2X+\tilde{X}^2, \quad \tilde{Y} = \tilde{X}^2-2\tilde{X},$$

and they enclose ($S_2$) the instabilities of growth rates $O(\epsilon^2)$ introduced by the shear. The discontinuous curve bounds the region $S_1$ of instabilities, also present in the corresponding sheet treatment, satisfying (3.40) so that $\tilde{Y} = \frac{1}{2}\tilde{X}^2-2$. The curve II touches the discontinuous curve at $\tilde{X} = 2$ so that the two curves form a cusp there.

possible that they occur in a shear containing *one* critical level, in which case $\delta$ is complex and $\tilde{\omega}_2$ is necessarily complex. Consequently the presence of a shear will tend either to amplify or to suppress these modes. Now if $X_1 = X_{1c+}$ then $X_3 = X_{1c+}+\alpha = -X_{1c-}$ while $X_1 = X_{1c-}$ implies that $X_3 = -X_{1c+}$. Thus if $\operatorname{Im}(\tilde{\omega}_2)$ is positive for one mode it will be negative for the other. Thus, provided $\tilde{\omega}_2$ is complex, one mode amplifies and the other decays. Now from (3.24) we see that $\overline{m}_1^{(0)}$ and $\overline{m}_3^{(0)}$ are either both real or both imaginary and from figure 2 we deduce that instability is possible only if both $\overline{m}_1^{(0)}$ and $\overline{m}_3^{(0)}$ are imaginary, and the solutions of the unstable mode decay exponentially as $|z|$ tends to infinity on either side. The condition for instability due to the presence of the shear layer can then be written as

$$|\min(X_{1c+}, X_{1c-})| < 1, \quad |\max(X_{1c+}, X_{1c-})| > (1+\beta^2)^{\frac{1}{2}},$$
$$1+|\alpha| > (1+\beta^2)^{\frac{1}{2}}, \quad 1+\tfrac{1}{2}\beta^2-\tfrac{1}{4}\alpha^2 > 0. \tag{3.37}$$

These inequalities can be shown to reduce to

$$|\alpha|^2-2|\alpha| < \beta^2 < |\alpha|^2+2|\alpha|. \tag{3.38}$$

In figure 3 we illustrate the region of instability (3.38) in the $(\tilde{X}, \tilde{Y})$ plane, where

$$\tilde{X} = |\alpha|, \quad \tilde{Y} = \beta^2, \tag{3.39}$$

to facilitate comparison with the other condition of instability

$$1+\tfrac{1}{2}\beta^2-\tfrac{1}{4}\alpha^2 < 0, \tag{3.40}$$

common to both layer and sheet models. It is immediately observed that the domains (3.38) and (3.40) do not intersect even if the last of (3.37) is discarded. We thus conclude that the presence of the shear introduces new modes of instability.

### 3.2. *Over-reflexion*

If we assume that $I \neq 0$ and (3.24) is *not* satisfied and carry out the above analysis we obtain

$$\frac{R_0}{I} = \frac{\overline{m}_1^{(0)}(1-X_1^2)-\overline{m}_3^{(0)}(1-X_3^2)}{\overline{m}_1^{(0)}(1-X_1^2)+\overline{m}_3^{(0)}(1-X_3^2)}, \quad \frac{T_0}{I} = \frac{2\overline{m}_1^{(0)}(1-X_1^2)\,X_3}{X_1[\overline{m}_1^{(0)}(1-X_1^2)+\overline{m}_3^{(0)}(1-X_3^2)]}, \tag{3.41a}$$

$$B_0 = 0, \quad A_0 = T_0/X_3, \quad R_1/I = i\overline{m}_1^{(0)}(1-X_1^2)\,\delta(R_0/I), \tag{3.41b}$$

while $B_1$ is given by the first of (3.29), and $T_1$ and $A_1$ can be obtained from the last two of (3.29) upon substituting for $R_1$. It must be noted that here $m_1$ is real so that $X_1$ is also real. Now to leading order both $R$ and $T$ (as represented by $R_0$ and $T_0$) are not influenced by the presence of critical levels and they take the same values as in the case of the corresponding sheet treatment. However, the order-$\epsilon$ corrections involve $\delta$ and according to the arguments advanced in the preceding subsection a net phase jump is present only if one critical level is present. Assuming for definiteness that $X_1 > 1$, the phase jump in $\delta$ is $-\pi$ if $|U_3| > |U_1|$ (see appendix). Also in this case $m_3$ must be imaginary (see figure 2) and $mk > 0$ (see figure 1). Thus to first order

$$|R/I| = 1+2\,|\epsilon\overline{m}_1^{(0)}(1-X_1^2)|\,\pi > 1 \tag{3.42}$$

and over-reflexion, though weak, takes place. Now the condition for (3.42) to hold for *some* $X_1$ is

$$1 < X_1^2 < 1+\beta^2, \quad X_3^2 < 1, \tag{3.43}$$

so that the over-reflecting régime lies outside the domain of instability as clearly shown by comparing (3.38) and (3.43). Thus the presence of *one* critical level effects both over-reflexion and instability but each phenomenon involves different modes. Moreover, over-reflexion of the sheet treatment (and the shear treatment containing *two* critical levels) involves the interaction of positive- and negative-energy waves while the critical level over-reflexion predicted here involves only positive-energy waves. It is already known, of course, that over-reflexion of positive-energy waves can exist even in the absence of critical levels (Eltayeb 1977) but the significance of the present situation is that over-reflexion occurs in a *stable* régime.

### 3.3. *Linear magnetic shear*

Before we conclude this section we shall examine the situation in which the flow is uniform everywhere and $V$ varies linearly from $V_1$ in region 1 to $V_3$ in region 3. Thus

$$U_0 = 0, \quad H(Z) = Z. \tag{3.44}$$

As a result of the continuity of $U$, the boundary conditions (2.10) simplify to

$$[w] = [w'] = 0. \tag{3.45}$$

It is also found that the quantity $Y$ defined by

$$Y = kV/\hat{\omega}, \quad \hat{\omega} = \omega_0 - kU_1, \tag{3.46}$$

can be used as the independent variable in place of $Z$ to reduce the present problem to a form similar to that discussed in the preceding two sections. In particular we have

$$g_0 = (1-Y^2)^{-2}, \quad g_1 = -2\overline{\omega}_1(1+Y^2)(1-Y^2)^{-3}, \tag{3.47a}$$

$$g_2 = \alpha_1^{-2}[-1+\beta_1^2/(1-Y^2)] + \overline{\omega}_1 W - 2\overline{\omega}_2(1+Y^2)(1-Y^2)^{-3}, \tag{3.47b}$$

where
$$\alpha_1 = kV_0/\hat{\omega}, \quad \beta_1^2 = N^2/\hat{\omega}^2, \quad \overline{\omega}_1 = \omega_1/\hat{\omega}, \quad \overline{\omega}_2 = \omega_2/\hat{\omega} \tag{3.48}$$

and $W$ is a complicated function of $Y$ and is not written down explicitly because it is not needed. Also the dispersion relation of the system assumes the form

$$\overline{m}^2 = -1 + \beta_1^2/(1-Y^2), \tag{3.49}$$

and is illustrated in figure 2(*b*).

The similarity of the problem to that discussed in §§ 3.1 and 3.2 permits us to omit the details. It should be remarked, however, that the continuity of $U$ makes the solution within the shear different. In particular the zeroth-order stability problem gives

$$B_0 = 0, \quad A_0 = T_0 = R_0, \quad w_0 = A_0, \tag{3.50}$$

so that the leading-order solution $w_0$ is a constant.

The first-order ($n = 1$) problem yields

$$\overline{\omega}_1 = 0, \quad \overline{m}_1^{(0)}(1-Y_1^2) + \overline{m}_3^{(0)}(1-Y_3^2) = 0. \tag{3.51}$$

Realizing that $Y$ is essentially the inverse of $X$, we recognize that (3.51) is identical to (3.24). If $U_1 \neq 0$ then provided $|V|$ increases through $|U_1|$ the results of §§ 3.1 and 3.2 hold good here. If, however, $U_1 = 0$ or $|U_1|$ does not lie in the range of $|V|$ then the roots relevant to this case are

$$\hat{\omega}_\pm = \pm\tfrac{1}{2}[N^2 + k^2(V_1^2 + V_3^2)]^{\frac{1}{2}}, \tag{3.52}$$

and instability in the presence of stable stratification (i.e. $N^2 > 0$) is not possible. Since (3.51) demands that $m_1/m_3$ be real we immediately (using the radiation condition) note that even neutral stability is not possible for real $m_1$ and $m_3$. When $m_1$, $m_3$ are both imaginary, on the other hand, neutral stability is possible in a small region of the $(\beta_1^2, |\alpha_1|)$ plane which can be shown to be

$$0 < |\alpha_1|^2 - 2 < \beta_1^2 < \min(1, 2|\alpha_1| - |\alpha_1|^2). \tag{3.53}$$

Now this domain occurs for a situation in which one critical level exists within the shear and the analysis of § 3.1 can easily be applied to this case to show that the presence of the critical level promotes instability by imparting a growth rate of $O(\epsilon^2)$ to the neutral modes (3.52) which satisfy (3.53).

The reflexion problem, solved along the lines of the analysis of § 3.2 above, yields

$$\frac{R_0}{I} = \frac{\overline{m}_1^{(0)}(1-Y_1^2) - \overline{m}_3^{(0)}(1-Y_3^2)}{\overline{m}_1^{(0)}(1-Y_1^2) + \overline{m}_3^{(0)}(1-Y_3^2)}, \quad \frac{T_0}{I} = \frac{A_0}{I} = \frac{2\overline{m}_1^{(0)}(1-Y_1^2)}{\overline{m}_1^{(0)}(1-Y_1^2) + \overline{m}_3^{(0)}(1-Y_3^2)}, \tag{3.54a}$$

$$\frac{R_1}{I} = i\overline{m}_1^{(0)}(1-Y_1^2)\,\delta\left(\frac{R_0}{I}\right), \quad \delta = \ln\left[\frac{(1+Y_3)(1-Y_1)}{(1-Y_3)(1+Y_1)}\right]. \tag{3.54b}$$

When $U_1 \neq 0$, the results deduced in § 3.2 for critical level over-reflexion hold good here provided $|V|$ increases through $|U_1|$. However, the most interesting situation here occurs when $U_1 = 0$ and no flow is present. Noting that propagation takes place only for

$$\max(0, 1-\beta_1^2) \leqslant Y^2 \leqslant 1, \tag{3.55}$$

so that, if we take $Y_1$ in this range and consider a situation in which $|V|$ increases from $|V_1|$ in region 1 to $|V_3|$ in region 3 in such a way that $|Y_3| > 1$, then the transmitted wave is evanescent and a critical level exists within the layer. The phase jump in $\delta$ is then $+\pi$ (see appendix) and

$$|R/I| = 1 - 2m_1 L(1-Y_1^2) + O(L^2) > 1 \tag{3.56}$$

since $m_1 < 0$ (see figure 1). Thus critical level over-reflexion occurs even in the absence of a flow.

## 4. Stability and over-reflexion by a smooth shear

In this section we shall investigate the stability and over-reflectivity of the smooth shear layer, i.e. the case in which $U$ and $V$ vary smoothly from $U_1$, $V_1$ in region 1 to $U_3$, $V_3$ in region 3 so that both $U'$ and $V'$ are continuous everywhere. The boundary conditions then take the form

$$[\phi] = [\phi'] = 0. \tag{4.1}$$

It turned out that the stability and reflectivity properties of the general shear are similar to those of the linear shear studied in the preceding section. The influence of the profiles of $U$ and $V$ on the stability is to modify the numerical value of the growth rate $|\mathrm{Im}(\omega_2)|$ while the order of magnitude remains the same (i.e. $\epsilon^2$). For this purpose no detailed calculations will be presented here but only a description of the first-order system is given in order to show the manner in which the solution adjusts itself to give the results similar to those of the linear shear despite the difference in the boundary conditions.

The leading-order problem ($n = 0$) is here governed by

$$\left.\begin{aligned} &\phi_0'' + g_0\phi_0 = 0, \\ &g_0 = \frac{k^2(V'^2 + VV'') + kU''\hat{\omega}_0}{\hat{\omega}_0^2 - k^2V^2} + \frac{k^3V(kVV'^2 + 2\hat{\omega}_0 V'U' + VU'^2)}{(\hat{\omega}_0^2 - k^2V^2)^2}, \end{aligned}\right\} \tag{4.2}$$

where $\phi_0$ must satisfy (4.1) above. Writing $\phi_0$ in the shear as

$$\phi_0 = A_0\phi_0^{(1)}(Z) + B_0\phi_0^{(2)}(Z), \tag{4.3}$$

it can be shown that

$$\phi_0^{(1)}(Z) = P_0^{\frac{1}{2}}\hat{\omega}_0, \quad \phi_0^{(2)}(Z) = \int^Z \frac{d\zeta}{[\phi_0^{(1)}(\zeta)]^2}, \tag{4.4}$$

in which

$$P_0 = -1 + k^2V^2/\hat{\omega}_0^2. \tag{4.5}$$

The application of the boundary conditions (4.1)–(4.3) then gives

$$R_0P_0^{\frac{1}{2}} = A_0\phi_0^{(1)}(0) + B_0\phi_0^{(2)}(0), \tag{4.6a}$$

$$T_0P_0^{\frac{1}{2}} = A_0\phi_0^{(1)}(1) + B_0\phi_0^{(2)}(1), \tag{4.6b}$$

$$0 = A_0\phi_0^{(1)\prime}(0) + B_0\phi_0^{(2)\prime}(1), \tag{4.6c}$$

$$0 = A_0\phi_0^{(1)\prime}(1) + B_0\phi_0^{(2)\prime}(1). \tag{4.6d}$$

Since $U'$ and $V'$ are both assumed to be continuous at $Z = 0, 1$, so that they vanish there, we see that

$$\phi_0^{(1)\prime}(0) = \phi_0^{(1)\prime}(1) = 0. \tag{4.7}$$

It then follows that

$$B_0 = 0, \quad T_0 = A_0\hat{\omega}_3, \quad R_0 = \hat{\omega}_1 T_0/\hat{\omega}_3, \tag{4.8}$$

in which

$$\hat{\omega}_1 = \omega_0 - kU_1 \quad \text{and} \quad \hat{\omega}_3 = \omega_0 - kU_3. \tag{4.9}$$

When we allow for the transformation (3.3) we see that (3.5) and (4.8) are identical. Consideration of higher-order terms leads to (3.23) and (3.24) and an expression similar to (3.35). The conditions for instability are then similar to those of the linear shear.

Over-reflexion by the smooth shear can also be shown to have the same properties as for the linear shear. The similarity between the linear and smooth shear can be explained in terms of the energy flux in the vertical direction (see § 5 below).

## 5. Concluding remarks

The stability of hydromagnetic–gravity waves in the presence of a thin layer (in the sense that $kL \ll 1$, where $k$ is the zonal wavenumber and $L$ the thickness of the shear layer) bounded by two uniform states on either side has been studied when the fluid is Boussinesq. In the special case in which the Alfvén speed is uniform (but necessarily non-zero) everywhere and the flow within the layer is linearly sheared, analytical results are obtained. In addition to the modes of instability and neutral stability present in the corresponding sheet (i.e. the case in which the flow experiences a sudden discontinuity) whose growth rates are here modified by an amount $O(\epsilon^2)$ the shear layer also possesses modes of instability whose growth rates are of the order $\epsilon^2 (= k^2L^2)$. These 'new' modes of instability are effected by the presence of *one* critical level within the shear and correspond to disturbances that decay on either side of the layer. Although the growth rates of these modes are only order $\epsilon^2$, nevertheless they may be more important than those also present in the sheet treatment since the new modes occur for smaller values of $|\alpha|$ (see figure 3 above). Thus for a given fluid (i.e. given $N$) and given uniform states in regions 1 and 3 the new modes of instability will be present while the others will not, if $|\alpha|$ is not too large. Furthermore, if the jump in the flow speed is increased, keeping everything else fixed, it would be expected that, by the time

$|\alpha|$ is large enough to make the mode $X_{1c\pm}$ (cf. (3.28)) unstable, the amplitude of the new mode of instability might acquire a large value. However, such a situation cannot be substantiated by the above linear analysis and a weakly nonlinear theory may provide some insight into the situation. Such a treatment is left to another publication.

Over-reflexion by the linear shear is also studied. It is shown that in addition to the over-reflecting régimes predicted by the corresponding sheet treatment other over-reflecting régimes are also present. These new regimes are again present only when one critical level lies within the shear. However, in contrast to the unstable modes mentioned above, these over-reflecting modes are associated with waves propagating in region 1 (i.e. $m_1$ real) and decaying exponentially in region 3 (i.e. $m_3$ imaginary). Now the relationship between over-reflecting and unstable (or neutrally stable) régimes is a subtle one. For a given situation (i.e. given fluid and basic state) the shear layer (or sheet) may be unstable to certain modes and over-reflecting to others. When considering over-reflexion, $m_1$ is assumed real and consequently $\omega$ and $k$ are also real. According to the linear normal mode treatment, only those *neutral* modes with real $m_1$ (and $m_3$) are obviously related to over-reflexion because resonance will occur. Other modes of instability or neutral stability will involve different values of $m_1$ (and $m_3$), and $k$ and although they may be present for such a state they are not obviously related to over-reflexion. On the other hand if the incident wave is viewed as a wave-packet it will involve a small frequency and zonal wave number bands and unless the over-reflecting modes involve frequencies *and* wavenumber swithin such bands no relationship between the two modes will exist, if nonlinear interactions are ruled out. With this in mind the new over-reflecting and unstable modes of the shear can occur independently.

In § 3.3 we discussed the linear magnetic shear to show, analytically, that both instability and over-reflexion are present *even if a flow is absent*, provided one critical level exists within the shear. Here also the unstable and over-reflecting régimes involve different modes. This case provides a marked contrast with the current sheet treatment in which instability (as opposed to neutral stability) and over-reflexion are both non-existent.

Both the linear magnetic and the linear velocity shear-layer models discussed in the preceding paragraphs of this section gave the normal mode equation predicted by the corresponding sheet treatment, in the leading order, and in search of identifying what type of thin shear will tally with the corresponding sheet in the limit of vanishing thickness we examined the general case of a magnetic-velocity shear in which both the flow and Alfvén speeds and their first derivatives are continuous everywhere (i.e. the smooth shear). Even in this general case the stability problem is found to be identical with that obtained for the linear shear to leading order. The growth rate of the new mode is also of order $\epsilon^2$ but the actual value depends on the profiles of $U$ and $V$. The stability and reflectivity of the general shear are in essence similar to those of the linear shear and the influence of the profile of the shear manifests itself only in a quantitative sense. It should be pointed out here, however, that this conclusion about the role of the shear profile may not hold good in the presence of all types of constraints (see, for example, Ahmed & Eltayeb 1980), particularly when the assumption $|\epsilon| \ll 1$ is relaxed.

Within the context of the present problem the similarity of the stability and reflectivity of the linear and smooth shear can be clarified by appealing to the total energy flux (i.e. the energy flux as measured by a stationary observer) in the vertical direction. This is given by

$$\mathscr{F} = -\tfrac{1}{4}i\rho_0\omega P(w^*dw/dz - w\,dw^*/dz)/k^2 \tag{5.1}$$

(cf. Acheson 1976, § 2; Eltayeb & McKenzie 1977, § 3), and is related to the wave-invariant (Eltayeb 1977, § 2) of the present system defined by

$$\mathscr{A} = \mathrm{Re}\,(-i\phi^*\phi'), \tag{5.2}$$

by the relation

$$\mathscr{F} = \tfrac{1}{2}\rho_0\omega\mathscr{A}/k^2. \tag{5.3}$$

Now the use of the boundary conditions (2.10) in (5.1) shows that $\mathscr{F}$ is continuous at $z = 0, L$ whether the shear is linear or smooth and by using (5.2) it follows that $\mathscr{F}$ is continuous and constant (for given $\omega$ and $k$) everywhere except at the critical levels, where it jumps from one constant value on one side to another constant value on the other side of the critical level. This can easily be seen by noting that if, $\mathscr{F}^{(i)}$ refers to region $i$, then

$$\mathscr{F}^{(1)} = \tfrac{1}{2}\rho_0\frac{\omega}{k^2}|R_0|^2\,\mathrm{Re}\,(m_1), \quad \mathscr{F}^{(2)} = -\tfrac{1}{4}\rho_0\frac{\omega}{k}\alpha\,\mathrm{Im}\,(A_0^*B_1),$$
$$\mathscr{F}^{(3)} = \tfrac{1}{2}\rho_0\frac{\omega}{k^2}|T_0|^2\,\mathrm{Re}\,(m_3), \tag{5.4}$$

to leading order. Thus any divergence of the shear results from those of the vortex sheet must be due to the influence of critical levels. Now the presence of two critical levels is found to yield a zero net phase jump and therefore the system behaves like the current-vortex sheet. In terms of the wave-normal curves of figure 1 the zero net phase jump can be interpreted in terms of the energy flux by noting that a wave approaching a critical level from the propagating side deposits some of its energy there to energe evanescent on the non-propagating side and remains evanescent until it encounters the other critical level where it picks up an amount of energy exactly equal to the amount lost at the first critical level to emerge as a propagating wave on the other side of the second critical level. In the case of the occurrence of *one* critical level the net phase jump is non-zero and corresponds to energy absorption or emission at the critical level, and it is the situation of energy emission that leads to instability or over-reflexion.

The numerical studies by Blumen *et al.* (1975) and Drazin & Davey (1977) on the stability of a shear layer of compressible isothermal fluid have shown that, in the presence of one critical level, one unstable mode of the shear reduces to that of the corresponding sheet in the limit of long waves and also new modes of instability wholly due to the presence of the shear are also present. These results are in general agreement with the analytical results obtained above for an incompressible fluid in the presence of both velocity and magnetic shear in the sense that the neutral mode $X_{1c+}$, as in (3.28), is destabilized by the presence of the shear and also because the new mode of instability has a growth rate that is much smaller than the growth rates of those present in the sheet treatment. However, it should be pointed out that the instabilities located here are due to the presence of the hydromagnetic critical levels which are predicted by a WKBJ treatment (see figure 1 above) and across which the total wave energy flux exhibits a discontinuity while the model of Blumen *et al.* (1975) contains a singularity whose presence is due to the presence of the shear and across which the total wave energy flux (which is proportional to the mean Reynolds stress $\tau$ exhibited in figure 6 of Blumen *et al.* (1975)) is continuous (Eltayeb 1977).

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## Appendix

Miles (1961) has shown that the solutions on either side of a critical level can be matched by simulating an initial-value problem. As a result the frequency $\omega$ is allowed to contain a small negative imaginary part $\omega_i$ so that the phase jump across the critical level can be determined. We shall apply this procedure to the quantity $\chi$ in (3.6) and (3.7). For definiteness suppose that $U(Z)$ is such that it increases from a value $U_1$ in region 1 to $U_3$ in region 3. Suppose further that $m_1$ is real, $k$ and $\omega$ are positive and $U_3$ is such that $\hat{\omega}_3^2 < -kV$. Such a wave crosses the critical level $k_{\infty+}$ and then the critical level $k_{\infty-}$. Let

$$\omega = \omega_0 + i\omega_i, \quad |\omega_i| < |\omega_0|, \quad \omega_i < 0. \tag{A 1}$$

In the neighbourhood of the critical level $k_{\infty+}$ the value $\chi_+$ of $\chi$ (occurring where $\hat{\omega}_0 = kV$) becomes

$$\chi_+ \simeq 2kV/[i\omega_i - kU_c'(Z-Z_c)], \tag{A 2}$$

where the subscript $c$ refers to quantities evaluated at the critical level $Z_c$. If

$$\arg(\chi_+) = \theta_1, \tag{A 3}$$

then

$$\tan\theta_1 = \frac{\omega_i/kU_c'}{(Z-Z_c)} \tag{A 4}$$

and since $\omega_i/kU_c' < 0$ then $\theta_1$ varies continuously from 0 to $+\pi$ as $Z$ varies from values much less than $Z_c$ to values much greater than $Z_c$.

In the vicinity of the critical level $k_{\infty-}$ we similarly have

$$\arg(\chi_-) = \theta_2, \quad \tan\theta_2 = \frac{-\omega_i/kU_c'}{(Z-Z_c)}, \tag{A 5}$$

and consequently the jump in $\theta_2$ as $Z$ increases through $Z$ is $-\pi$.

It may then be concluded that $\arg(\chi_1) = -\arg(\chi_3)$ and the net jump in phase across the two critical levels is zero. Similar arguments apply to all other cases in which two critical levels lie within the layer.

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